Integrating factors

Most first order equations are not exact but if an equation is exact then we can solve it. This leads us to ask: if the equation is not exact, can we make it exact? The answer is yes, under certain circumstances.

Recall –

If the equation $M(x,y)dx + N(x,y)dy = 0$ is exact, that is if

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then there exists $F(x,y) = C$ such that

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$, and $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = Mdx + Ndy = 0$

so $F(x,y) = C$ satisfies the given equation and your job is to find F.

Now suppose the equation $M(x,y)dx + N(x,y)dy = 0$ <u>is not exact</u>.

Furthermore, suppose that, either by skill or by luck, we can find a function $\mu(x,y)$ such that if we multiply our equation by μ then the equivalent equation $\mu M dx + \mu N dy = 0$ is exact. Then we can solve this new equation and, since the equations are equivalent, we have a solution of the original equation $Mdx + Ndy = 0$.

<u>ex</u>: $(y^2 + xy)dx - x^2 dy = 0$ is not exact because

$$\frac{\partial}{\partial y}(y^2 + xy) = 2y + x$$

but $\frac{\partial}{\partial x}(-x^2) = -2x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact. ☹

Now we let $\mu(x,y) = \frac{1}{xy^2}$. Don't worry, I pulled this out of thin air.

If we multiply the original equation by μ we find –

$$\frac{1}{xy^2}(y^2 + xy)dx + \frac{1}{xy^2}(-x^2)dy = 0$$

$$\left(\frac{1}{x} + \frac{1}{y}\right)dx - \frac{x}{y^2}dy = 0$$

Checking for exactness we find

$$\frac{\partial}{\partial y}\left(\frac{1}{x} + \frac{1}{y}\right) = -\frac{1}{y^2} = \frac{\partial}{\partial x}\left(\frac{-x}{y^2}\right)$$

→ So this new equation is exact and we can solve it.
Since this equation is equivalent to the original equation we have our solution. So let's solve it.

Since $\left(\frac{1}{x}+\frac{1}{y}\right)dx - \frac{x}{y^2}dy = 0$ is exact there exists a function $F(x,y)=C$ such that $\frac{\partial F}{\partial x} = \left(\frac{1}{x}+\frac{1}{y}\right)$, $\frac{dF}{\partial y} = -\frac{x}{y^2}$, and

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \left(\frac{1}{x}+\frac{1}{y}\right)dx - \frac{x}{y^2}dy = 0$$ and $F = C$ is the solution we seek. You can begin with either $\frac{\partial F}{\partial x}$ or $\frac{\partial F}{\partial y}$, let's say

$$\frac{\partial F}{\partial x} = \frac{1}{x}+\frac{1}{y}$$

$$F = \int\left(\frac{1}{x}+\frac{1}{y}\right)dx$$

$$= \ln|x| + \frac{x}{y} + G(y)$$

Then $\frac{\partial F}{\partial y} = -\frac{x}{y^2} + G'(y) = -\frac{x}{y^2}$

so $G'(y)=0$, $G(y) = C_1$, and

$$F(x,y) = \ln|x| + \frac{x}{y} + C_1 = C$$

or $\ln|x| + \frac{x}{y} = C$ is the general solution we seek.

We can verify this is a solution to our original equation by differentiating with respect to x —

$$\frac{d}{dx}\left\{\ln|x| + \frac{x}{y}\right\} = \frac{d}{dx}(C)$$

$$\frac{1}{x} + \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$$

$$y^2 + xy - x^2\frac{dy}{dx} = 0$$

or $(y^2+xy)dx - x^2dy = 0$

which is our original equation!

So the solution of

$$(y^2+xy)dx - x^2dy = 0$$

is $\boxed{\ln|x| + \frac{x}{y} = C}$, $x \neq 0, y \neq 0$

In the previous problem I just gave the function $\mu(x,y) = \frac{1}{xy^2}$.

This function μ is called an integrating factor.

Let's find the conditions under which we can find μ.

We have an equation $M(x,y)dx + N(x,y)dy = 0$ which is not exact. We seek a function $\mu(x,y)$ such that

$\mu M dx + \mu N dy = 0$ is exact.

In order to be exact it must be true that

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \quad \text{or}$$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

This is a PDE for μ which is generally impossible to solve.

Maybe we can find $\mu = \mu(x)$ which is a function of one variable only and we have a better chance of finding it. In this case $\mu = \mu(x)$, $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$, and $\frac{\partial \mu}{\partial y} = 0$. Then we have

$$\mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{d\mu}{dx}$$

and $$\frac{d\mu}{dx} = \mu \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right)$$

This is still a problem because the RHS is still a function of both x and y.

If however $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone then

$$\frac{d\mu}{\mu} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx,$$ the equation is separable, and

$$\int \frac{d\mu}{\mu} = \int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx$$

So we can solve for $\mu(x)$. Of course, these conditions are incredibly restrictive and the odds that

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone are pretty small.

So, in general, for some arbitrary 1st order ODE, we probably can't find an integrating factor.

1^{st} order linear equations

A first order linear equation always has an integrating factor $\mu(x)$. Fortunately, a lot of the most commonly occuring equations in applications are linear.

A first order linear ODE is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Of course, if $Q(x)=0$ this equation is separable. You should also check for exactness before you do more work. That is if

$(Py-Q)dx + dy$ is exact then

$$\frac{\partial}{\partial y}(Py-Q) = P = \frac{\partial}{\partial x}(1) = 0$$

So the only case where this equation is exact is where $P(x)=0$ and $\frac{dy}{dx} = Q(x)$ which is trivially integrable. (Thanks Kennard).

So let's suppose we have $\frac{dy}{dx} + P(x)y = Q(x)$ where $P, Q \neq 0$.

In differential form we have

$$(Py-Q)dx + dy = 0$$

In what follows remember $P=P(x)$, $Q=Q(x)$.

We seek an integrating factor $\mu(x)$ such that

$\mu(Py-Q)dx + \mu dy = 0$ is exact.

If this is true then

$$\frac{\partial}{\partial y}(\mu Py - \mu Q) = \mu P = \frac{\partial}{\partial x}\mu = \frac{d\mu}{dx}$$ (because $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$ since $\mu = \mu(x)$)

So we have a separable equation in μ –

$$\mu P = \frac{d\mu}{dx}$$

$$\frac{d\mu}{\mu} = P dx$$

$$\int \frac{d\mu}{\mu} = \int P dx$$

$$\ln|\mu| = \int P dx$$

or $\ln \mu = \int P dx$ if we assume $\mu > 0$

Then $\mu = e^{\int P dx}$ is an integrating factor.

2 notes are in order –

1st assumed $\mu > 0$ which is justified by the result $\mu = e^{\int P dx}$ which is in fact always positive. But notice $\mu = -e^{\int P dx}$ is also an integrating factor.

Second, note that if $\mu = e^{\int P dx}$ is an integrating factor then

$$\mu = e^{\int P dx + c} = e^c e^{\int P dx} = Ce^{\int P dx}$$ is also an integrating factor.

So there are, in fact, an infinite number of integrating factors. We only need one so we take the simplest, $\mu = e^{\int P dx}$.

So, to summarize, given the 1st order linear ODE,

$$\frac{dy}{dx} + P(x)y = Q(x)$$ or, in differential form,

$$(P(x)y - Q(x))dx + dy = 0$$

we can find an integrating factor $\mu(x) = e^{\int P(x)dx}$ such that

$$\mu \frac{dy}{dx} + \mu P y = \mu Q$$ or

$(\mu P y - \mu Q)dx + \mu dy = 0$ is exact, which you can verify –

$$\frac{\partial}{\partial y}(\mu P y - \mu Q) = \mu P$$

and $\frac{\partial}{\partial x}\mu = \frac{d}{dx}\mu = \frac{d}{dx} e^{\int P(x)dx} = e^{\int P dx} \frac{d}{dx}\int P(x)dx = \mu P$.

So we can now solve the exact equation using the method we already know for exact equations.

As an alternative notice that

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu \frac{dy}{dx} + y\frac{d\mu}{dx} \\
&= \mu \frac{dy}{dx} + y\frac{d}{dx}e^{\int P dx} \\
&= \mu \frac{dy}{dx} + ye^{\int P dx} \cdot \frac{d}{dx}\int P dx \\
&= \mu \frac{dy}{dx} + ye^{\int P dx} \cdot P \\
&= \mu \frac{dy}{dx} + \mu P y
\end{aligned}$$

which is the LHS of the equation $\mu \frac{dy}{dx} + \mu P y = \mu Q$ which we want to solve.

So we don't really need to go through the process of solving the exact equation. We have

$$\mu \frac{dy}{dx} + \mu P y = \mu Q$$

$$\frac{d}{dx}(\mu y) = \mu Q$$

$$\int \frac{d}{dx}(\mu y)dx = \int \mu Q dx$$

$$\mu y = \int \mu Q dx$$

and $$y = \frac{1}{\mu}\int \mu Q dx$$

is the solution we seek. Seems easier to me but you can use either method. You choose.

Ex: Solve the IVP $y' - 2xy = x$, $y(0) = 0$

This is a 1^{st} order linear equation of the form $y' + Py = Q$ and it is not exact as you can show —

If $$\frac{dy}{dx} - 2xy = x$$

then $$(2xy + x)dx - dy = 0$$

$$\frac{\partial}{\partial y}(2xy + x) = 2x$$

but $\frac{\partial}{\partial x}(-1) = 0$ so the equation is not exact.

Identifying $P(x) = -2x$ we find an integrating factor

$$\mu = e^{\int P(x)dx} = e^{-\int 2x dx} = e^{-x^2}$$

Multiplying by μ we find

$$(2xye^{-x^2} + xe^{-x^2})dx - e^{-x^2}dy = 0$$

Check that this is exact —

$$\frac{\partial}{\partial y}(2xye^{-x^2} + xe^{-x^2}) = 2xe^{-x^2}$$

and $\frac{\partial}{\partial x}(-e^{-x^2}) = 2xe^{-x^2}$ so the equation is exact.

So there exists a function $F(x,y) = C$ such that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = (2xye^{-x^2} + xe^{-x^2})dx - e^{-x^2}dy = 0$$

so $$\frac{\partial F}{\partial x} = (2xye^{-x^2} + xe^{-x^2}) = xe^{-x^2}(2y+1)$$

$$F = (2y+1)\left(-\frac{1}{2}\right)e^{-x^2} + G(y)$$

Then $\frac{\partial F}{\partial y} = -e^{-x^2} + G'(y) = -e^{-x^2}$

So $G'(y) = 0$

$G(y) = C = \text{constant}$

and $F(x,y) = -\frac{1}{2}e^{-x^2}(2y+1) + C = C$

or $-\frac{1}{2}e^{-x^2}(2y+1) = C$ is the general solution

applying the condition $y(0)=0$ we find

$C = \frac{-1}{2}$

and $e^{-x^2}(2y+1) = 1$ is the solution. We can solve explicitly for y and we find

$\boxed{y = \frac{1}{2}e^{x^2} - \frac{1}{2}}$ is the solution of the IVP.

As exercise, you should show that this expression satisfies both the ODE $\frac{dy}{dx} - 2xy = x$ and the IC $y(0)=0$.

As an alternative, we can solve using the new method above. Since $\frac{dy}{dx} - 2xy = x$ is linear we find an integrating factor

$$\mu = e^{-\int 2x\,dx} = e^{-x^2}$$

and note that $\frac{d}{dx}(\mu y) = \frac{d}{dx}(ye^{-x^2}) = e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2}$ which is the LHS of our equation multiplied by e^{-x^2}. So

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = xe^{-x^2}$$

$$\frac{d}{dx}(ye^{-x^2}) = xe^{-x^2}$$

$$ye^{-x^2} = \int xe^{-x^2}dx$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$y = Ce^{x^2} - \frac{1}{2}$$

But $y(0) = 0 = C - \frac{1}{2}$ so $C = \frac{1}{2}$ and

$\boxed{y = \frac{1}{2}e^{x^2} - \frac{1}{2}}$ which is the same solution!

It's your choice, I find the second method a bit nicer.